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On the Quasi-TEM Modes in Inhomogeneous Multiconductor Transmission Lines

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***Abstract*— We consider the general inhomogeneous shielded N-conductor transmission line and derive several properties for the quasi-TEM modes. The concept of quasi-TEM is deduced through an asymptotic series expansion of the fields and conditions for the propagation constant as well as the construction of the field are presented. It is seen that the problem is reduced to two static two-dimensional boundary value problems. The concepts of propagating modes and impedance modes are introduced and it is shown, that in the general case, these are not the same. The special cases of propagating impedance modes are finally discussed and are seen to exist under certain symmetry conditions of the multiconductor line.**

I. Introduction

THE INHOMOGENEOUS, uniform, multiconductor transmission line is a popular component in many microwave applications, especially in filter-directional coupler design. It is also well known that the dominant waves at the low-frequency end of the spectrum are not of pure TEM but quasi-TEM character. For a one-conductor shielded line this has been confirmed through asymptotic field analysis in [1], [2], but a solid systematic theory for multiconductor lines seems to be lacking. In what follows, the Sections II–V give the analysis and the method of constructing the quasi-TEM fields for a general inhomogeneous N-conductor shielded line. The construction is based on solutions of two sets of static field problems plus an eigenvalue problem for boundary conditions of the propagating modes.

The boundary condition or circuit theoretical point of view is then treated in Sections VI–IX. Previous considerations [5], [6], based on the assumed quasi-TEM character of the fields, have concentrated only on propagating modes. Being simpler at the terminations of the line, another set of modes, impedance modes, are introduced here and their relation to the propagating modes is studied. The impedance modes are defined as such voltage and current distributions on the line that are the same except for a scalar

Manuscript received October 14, 1980; revised February 27, 1981.

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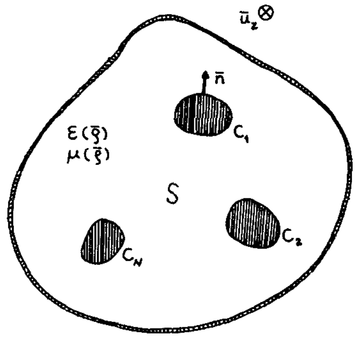

Fig. 1. Cross section of an inhomogeneous shielded N-conductor line.

factor. These modes do not proceed independently of each other, but, instead, they are coupled as they propagate. Finally, the special and interesting case of propagating impedance modes are discussed and are found to exist under symmetry conditions of the line.

II. The Inhomogeneous Multiconductor Line

We consider the problem of a shielded N-conductor line, uniform in the z direction and filled with an isotropic material, which may be inhomogeneous in the transverse plane $\epsilon=\epsilon(\bar{\rho})$, $\mu=\mu(\bar{\rho})$. (See Fig. 1.) The fields in the waveguide satisfy Maxwell's equations. Assuming a propagating wave solution $\bar{E}(\bar{\rho})e^{-j\beta z}$, $\bar{H}(\bar{\rho})e^{-j\beta z}$, existing in any structure with a translatoric invariance in the z direction, we can write

$$\left.\begin{aligned} \nabla\times\bar{E}&=j\beta\bar{u}_z\times\bar{E}-j\omega\mu\bar{H} \quad (1)\\ \nabla\times\bar{H}&=j\beta\bar{u}_z\times\bar{H}+j\omega\epsilon\bar{E} \quad (2)\end{aligned}\right\}\text{on } S.$$

The boundary conditions on the conductor surfaces are

$$\left.\begin{aligned} \bar{n}\times\bar{E}&=0 \quad (3)\\ \bar{n}\times(\nabla\times\bar{H})&=j\beta\bar{u}_z(\bar{n}\cdot\bar{H})=0 \quad (4)\end{aligned}\right\}\text{on } C_i.$$

If we write the field vectors in transversal and longitudinal components $\bar{E}(\bar{\rho})=\bar{e}(\bar{\rho})+\bar{u}_z e_z(\bar{\rho})$, $\bar{H}(\bar{\rho})=\bar{h}(\bar{\rho})+\bar{u}_z h_z(\bar{\rho})$, we have from (1) to (4)

$$\left.\begin{aligned} \nabla\times\bar{e}&=-j\omega\mu\bar{u}_z h_z \quad (5)\\ \nabla e_z&=-j\beta\bar{e}-j\omega\mu\bar{u}_z\times\bar{h} \quad (6)\\ \nabla\times\bar{h}&=j\omega\epsilon\bar{u}_z e_z \quad (7)\\ \nabla h_z&=-j\beta\bar{h}+j\omega\epsilon\bar{u}_z\times\bar{e} \quad (8)\end{aligned}\right\}\text{on } S$$

$$\left.\begin{aligned} \bar{n}\times\bar{e}&=0 \quad (9)\\ e_z&=0 \quad (10)\\ \bar{n}\times(\nabla\times\bar{h})&=0 \quad (11)\\ \bar{n}\cdot\nabla h_z&=-j\beta\bar{n}\cdot\bar{h}=0 \quad (12)\end{aligned}\right\}\text{on } C_i.$$

III. Asymptotic Series Solution

When looking for a quasi-TEM wave in the inhomogeneous line we must start by considering an asymptotic series solution for the equations (5)–(12), valid for $\omega\to 0$. In fact, we might write a series expansion for every quantity on ω, e.g., $\bar{e}(\omega)=\Sigma\bar{e}_n\omega^n$. It is instructive to study first the dependence of different quantities on ω. The complex field vectors can be thought to arise from real time-dependent vectors through the Fourier transform

$$\bar{F}(\omega)=\int_{-\infty}^{\infty}\bar{F}(t)e^{-j\omega t}d\omega. \quad (13)$$

Because of this, we have the property $\bar{F}^*(-\omega)=\bar{F}(\omega)$, or the real part of all field vectors is an even function and the imaginary part an odd function of ω. In fact, separating the z-dependent term $e^{-j\beta(\omega)z}$, we see that $\beta^*(-\omega)=-\beta(\omega)$, or the propagation is an odd function, whereas the attenuation factor is an even function of ω. For a propagating wave, $\bar{e}$ and $\bar{h}$ are real functions, whereas from (5) and (7) we see that e_z and h_z are imaginary. In this case, $\bar{e}$ and $\bar{h}$ are even functions and e_z, h_z odd functions of ω, the result given in [1]. If the line has losses, the picture is more complicated. We however assume here the absence of losses because of simplicity, whence the asymptotic series can be written as

$$\beta(\omega)=\beta_1\omega+\beta_3\omega^3+\cdots \quad (14)$$

$$\bar{e}(\omega)=\bar{e}_0+\bar{e}_2\omega^2+\cdots \quad (15)$$

$$e_z(\omega)=e_1\omega+e_3\omega^3+\cdots \quad (16)$$

$$\bar{h}(\omega)=\bar{h}_0+\bar{h}_2\omega^2+\cdots \quad (17)$$

$$h_z(\omega)=h_1\omega+h_3\omega^3+\cdots. \quad (18)$$

These series are asymptotic in the sense that the smaller ω, the better the approximation by the first few terms. The field $\bar{e}_0$, $\bar{h}_0$, $\bar{e}_1$, $\bar{h}_1$ is called the quasi-TEM field.

If the series (14)–(18) are substituted in (5)–(12), and in each equation the coefficients of the different powers of ω are equated separately, we are left with a set of equations for the unknown coefficients. The first few of these are

$$\left.\begin{aligned} \nabla\times\bar{e}_0&=0 \quad (19)\\ \nabla e_1&=-j\beta_1\bar{e}_0-j\mu\bar{u}_z\times\bar{h}_0 \quad (20)\\ \nabla\times\bar{h}_0&=0 \quad (21)\\ \nabla h_1&=-j\beta_1\bar{h}_0+j\epsilon\bar{u}_z\times\bar{e}_0 \quad (22)\end{aligned}\right\}\text{on } S$$

$$\left.\begin{aligned} \bar{n}\times\bar{e}_0&=0 \quad (23)\\ e_1&=0 \quad (24)\\ \bar{n}\times(\nabla\times\bar{h}_0)&=0 \quad (25)\\ \bar{n}\cdot\bar{h}_0&=0 \quad (26)\\ \bar{n}\cdot\nabla h_1&=0 \quad (27)\end{aligned}\right\}\text{on } C_i.$$

These equations will suffice for our purposes. It is seen that (25) is superfluous since it is satisfied if (21) is.

IV. The Static Field Problem

The equations governing the zeroth-order fields $\bar{e}_0$, $\bar{h}_0$ are (19), (21), which are not enough. Taking the curl operation of (20) and (22) gives us

$$\nabla\cdot(\epsilon\bar{e}_0)=0 \quad (28)$$

$$\nabla\cdot(\mu\bar{h}_0)=0 \quad (29)$$

after which we have a sufficient set of equations. These equations are of the static type and the solution can be effectuated by use of potential functions. In fact, we may look for a scalar potential $\phi(\bar{\rho})$ giving the electric field as $\bar{e} = -\nabla\phi$. Then, (19) is satisfied identically and from (28) we have

$$\nabla\cdot(\epsilon\nabla\phi)=0 \tag{30}$$

which is the generalized Laplace equation. Likewise, we may look for a vector potential $\bar{A}(\bar{\rho})$ for the magnetic field: $\bar{h}_0 = 1/\mu\,\nabla\times\bar{A}$, because (29) will be satisfied identically. Since $\bar{h}_0$ is a transversal vector, $\bar{A}$ must be longitudinal, i.e., $\bar{A}=\bar{u}_z A(\bar{\rho})$, and the equation for this potential is obtained from (21)

$$\nabla\cdot\left(\frac{1}{\mu}\nabla A\right)=0 \tag{31}$$

which, also, is a generalized Laplace equation. As boundary conditions for the potentials we obtain from (23) and (26)

$$\bar{n}\times\nabla\phi=0 \text{ on } C_i \text{ or } \phi=U_i \text{ on } C_i \tag{32}$$

$$\bar{n}\times\nabla A=0 \text{ on } C_i \text{ or } A=\psi_i \text{ on } C_i. \tag{33}$$

Thus the potentials ϕ, A assume constant values on each of the conductor surfaces. Because the reference point of the potentials is immaterial, we may take the value of the potentials to be zero on the sheath.

The problems for ϕ and A are now completely specified and the solution functions $\phi(\bar{\rho})$, $A(\bar{\rho})$ depend uniquely on the set of boundary values $\{U_i\}$, $\{\psi_i\}$. U_i is the voltage between the conductor i and the sheath, whereas ψ_i can easily be seen to represent the magnetic flux between the conductor i and the sheath. In fact, we may write $\psi_i = \int\mu\bar{h}_0\cdot(\bar{u}_z\times d\bar{c}) = -\int\nabla A\cdot d\bar{c} = A(C_i)$ integrating along a curve from the conductor i to the sheath. Until now there is no connection between the electrostatic field $\bar{e}_0$ and the magnetostatic field $\bar{h}_0$, because their potential problems are related in no way.

V. The Quasi-TEM Field

The fields $\bar{e}_0$, $\bar{h}_0$ are related through (20) and (21). In fact, in order to satisfy the boundary equation (24), the line integral of ∇e_1 from any conductor i to the sheath must be zero. Hence, from (20) there arises an integral relation between $\bar{e}_0$ and $\bar{h}_0$

$$\int_i^0 \nabla A\cdot d\bar{c} = \beta_1\int_i^0\nabla\phi\cdot d\bar{c} \text{ or } \psi_i=\beta_1 U_i. \tag{34}$$

Also, to be a physical quantity, h_1 must be unique, i.e., any closed integral of ∇h_1 must give us zero. Integrating along a curve around the conductor C_i, we have another relation between $\bar{e}_0$ and $\bar{h}_0$ from (22)

$$\int_{C_i}\epsilon\bar{e}_0\cdot\bar{n}\,dc-\beta_1\int_{C_i}h_0\cdot d\bar{c}=Q_i-\beta_1 I_i=0. \tag{35}$$

Here, Q_i, is the charge per unit length in the electrostatic problem and I_i is the static current in the magnetostatic problem.

In the electrostatic problem we have a linear relation between the quantities $\{Q_i\}$ and $\{U_i\}$; in fact, written in matrix form, we have

$$\underline{Q}=\underline{\underline{C}}\cdot\underline{U} \tag{36}$$

where $\underline{\underline{C}}$ is the static capacitance-per-unit-length matrix. Correspondingly, from the magnetostatic problem we may calculate the inductance-per-unit-length matrix $\underline{\underline{L}}$ defined by

$$\underline{\psi}=\underline{\underline{L}}\cdot\underline{I}. \tag{37}$$

The propagation constant of a quasi-TEM wave is now seen to satisfy both $\underline{\psi}=\beta_1\underline{U}$ and $\underline{Q}=\beta_1\underline{I}$, whence we have an eigenvalue equation for β_1

$$\underline{\underline{L}}\cdot\underline{\underline{C}}\cdot\underline{U}=\beta_1^2\underline{U}. \tag{38}$$

Hence, for a quasi-TEM wave the voltages on the conductors C_i may not be chosen arbitrarily, but they must be an eigenvector of the matrix $\underline{\underline{L}}\cdot\underline{\underline{C}}$. Correspondingly, the magnetic fluxes of the conductors must be eigenvectors of the same matrix, i.e., the fluxes and the voltages must be in the same ratio $\psi_i/U_i=\psi_j/U_j=\beta_1$. Moreover, the propagation factor is an eigenvalue of the matrix $(\underline{\underline{L}}\cdot\underline{\underline{C}})^{1/2}$.

In the general case there exist N different eigenvectors, i.e., voltage distributions corresponding to N different propagation factors $\beta_1\omega$.

The longitudinal components of the quasi-TEM wave e_1, h_1 are obtained from (20) and (22). In fact, from (20) we have

$$\nabla e_1=j\beta_1\nabla\phi-j\bar{u}_z\times(\nabla A\times\bar{u}_z)=\nabla(j\beta_1\phi-jA) \tag{39}$$

or

$$e_1=j(\beta_1\phi-A) \tag{40}$$

because due to (34), the boundary condition (24) is obviously satisfied. The other field h_1 cannot be written in an explicit form

$$\nabla h_1=j\bar{u}_z\times\left(\beta_1\frac{1}{\mu}\nabla A-\epsilon\nabla\phi\right) \tag{41}$$

because the right-hand side cannot be written as a gradient of an explicit scalar function, although its curl vanishes. Also, the boundary condition (27) is satisfied. Of course, the value of h_1 at any point can be obtained by integrating the right-hand side of (41) to that point. A simple expression for h_1 would arise, if we would solve for the dual static potentials, i.e., setting $e_0=1/\epsilon\,\nabla B\times\bar{u}_z$ and $\bar{h}_0=-\nabla\eta$, whence $h_1=j\beta_1\eta+jB$. The quasi-TEM approximation for the electromagnetic field in the multiconductor line can be written as

$$\bar{E}(\bar{\rho})=\bar{e}_0(\bar{\rho})+\bar{u}_z\omega e_1(\bar{\rho}) \tag{42}$$

$$\bar{H}(\bar{\rho})=\bar{h}_0(\bar{\rho})+\bar{u}_z\omega h_1(\bar{\rho}) \tag{43}$$

$$\beta=\omega\beta_1. \tag{44}$$

As the frequency is decreased, the field is seen to become more TEM. Also, if the medium is homogeneous, or inhomogeneous in such a way that $\epsilon\mu$ is constant, the potentials are seen to relate as $A(\bar{\rho})=\beta_1\phi(\bar{\rho})$, and $\beta_1=\sqrt{\mu\epsilon}$, whence

$e_1=0$ and $h_1=0$, and the field is exactly TEM. In this case, the $\underline{\underline{C}}$ and $\underline{\underline{L}}$ matrices are related as $\underline{\underline{L}}\cdot\underline{\underline{C}}=\mu\epsilon\underline{\underline{I}}$.

VI. The General Quasi-TEM Field

In the previous analysis, one of the propagating quasi-TEM modes was considered. For a N-conductor shielded line there exist N such modes, each with a voltage vector $\underline{U}^i$ and a current vector $\underline{I}^i$, $i=1, 2,\cdots,N$. Further, for each mode there exists a propagation factor β^i, potential functions $\phi^i(\bar{\rho})$, $A^i(\bar{\rho})$, and finally, the fields $\bar{e}_0^i(\bar{\rho})$, $\bar{h}_0^i(\bar{\rho})$, $e_1^i(\bar{\rho})$, $h_1^i(\bar{\rho})$.

Now we start to consider the most general combination of these modes, i.e., voltage vector $\underline{U}$ and current vector $\underline{I}$. These are functions of the coordinate z. From Faraday's and Ampere's laws we have the following transmission-line equations:

$$\underline{U}'(z)=-j\omega\underline{\psi}=-j\omega\underline{\underline{L}}\cdot\underline{I}(z) \tag{45}$$

$$\underline{I}'(z)=-j\omega\underline{Q}=-j\omega\underline{\underline{C}}\cdot\underline{U}(z), \qquad '=d/dz. \tag{46}$$

By elimination, the second-degree equations result

$$\underline{U}''(z)=-\omega^2\underline{\underline{L}}\cdot\underline{\underline{C}}\cdot\underline{U}(z) \tag{47}$$

$$\underline{I}''(z)=-\omega^2\underline{\underline{C}}\cdot\underline{\underline{L}}\cdot\underline{I}(z). \tag{48}$$

Note that the operator $(d^2/dz^2)\underline{\underline{I}}+\omega^2\underline{\underline{L}}\cdot\underline{\underline{C}}$ can be written as $[(d/dz)\underline{\underline{I}}+j\omega(\underline{\underline{L}}\cdot\underline{\underline{C}})^{1/2}]\cdot[(d/dz)\underline{\underline{I}}-j\omega(\underline{\underline{L}}\cdot\underline{\underline{C}})^{1/2}]$. Since we only consider solutions propagating in the positive z direction, instead of (47) and (48) we may write

$$\underline{U}'(z)=-j\omega(\underline{\underline{L}}\cdot\underline{\underline{C}})^{1/2}\cdot\underline{U}(z) \tag{49}$$

$$\underline{I}'(z)=-j\omega(\underline{\underline{C}}\cdot\underline{\underline{L}})^{1/2}\cdot\underline{I}(z). \tag{50}$$

The square-roots of the matrices are taken to be positive definite since the matrices $\underline{\underline{L}},\underline{\underline{C}}$ are positive definite for lossless lines. Moreover, $\underline{\underline{L}}$ and $\underline{\underline{C}}$ are symmetric for isotropic media, but the square roots are only symmetric if the matrices $\underline{\underline{L}}$ and $\underline{\underline{C}}$ commute, whence they possess the same set of eigenvectors. It is exactly in this case, that we can write $\underline{\underline{L}}^{1/2}\cdot\underline{\underline{C}}^{1/2}$ for $(\underline{\underline{L}}\cdot\underline{\underline{C}})^{1/2}$ and $(\underline{\underline{C}}\cdot\underline{\underline{L}})^{1/2}$, which can be verified with some effort.

The most general solutions for (49) and (50) can be written

$$\underline{U}(z)=e^{-j\omega(\underline{\underline{L}}\cdot\underline{\underline{C}})^{1/2}z}\cdot\underline{U}(0) \tag{51}$$

$$\underline{I}(z)=e^{-j\omega(\underline{\underline{C}}\cdot\underline{\underline{L}})^{1/2}z}\cdot\underline{I}(0) \tag{52}$$

where the matrix exponential has the meaning

$$e^{\underline{\underline{A}}}=\underline{\underline{I}}+\underline{\underline{A}}+\frac{1}{2!}\underline{\underline{A}}^2+\cdots. \tag{53}$$

Since $d/dz(e^{\underline{\underline{A}}z})=\underline{\underline{A}}\cdot(e^{\underline{\underline{A}}z})$, (51) and (52) are solutions of (49) and (50) for any $\underline{U}(0)$, $\underline{I}(0)$, and $\underline{U}(z)$, $\underline{I}(z)$ can be interpreted as voltage and current waves propagating with a matrix propagation factor.

Because the set of eigenvectors of the matrices $\underline{\underline{L}}\cdot\underline{\underline{C}}$, $(\underline{\underline{L}}\cdot\underline{\underline{C}})^{1/2}$ and $\exp(-j\omega(\underline{\underline{L}}\cdot\underline{\underline{C}})^{1/2})$ is the same, the eigenvalues corresponding to the eigenvector $\underline{U}^i$ of (38) are $(\beta_1^i)^2$ and $\exp(-j\omega\beta_1^i)$, correspondingly. An eigenvector decomposition of $\underline{U}(0)$ in (51) gives us a propagating quasi-TEM mode decomposition of the general voltage vector $\underline{U}(z)$.

VII. Impedance Quantities of Quasi-TEM Modes

The impedance and admittance matrices of the multiconductor line express the linear connection between the voltage and current vectors

$$\underline{U}=\underline{\underline{Z}}\cdot\underline{I} \qquad \underline{I}=\underline{\underline{Y}}\cdot\underline{U}, \qquad \underline{\underline{Z}}=\underline{\underline{Y}}^{-1}. \tag{54}$$

Differentiating these equations and substituting in (45), (46), (49), and (50) gives us the relations between the matrices $\underline{\underline{Z}}$, $\underline{\underline{L}}$, $\underline{\underline{C}}$

$$\underline{\underline{L}}=\underline{\underline{Z}}\cdot\underline{\underline{C}}\cdot\underline{\underline{Z}} \quad \text{or} \quad \underline{\underline{C}}=\underline{\underline{Y}}\cdot\underline{\underline{L}}\cdot\underline{\underline{Y}} \tag{55}$$

and hence

$$\underline{\underline{Z}}=(\underline{\underline{L}}\cdot\underline{\underline{C}})^{1/2}\cdot\underline{\underline{C}}^{-1}=\underline{\underline{C}}^{-1}\cdot(\underline{\underline{C}}\cdot\underline{\underline{L}})^{1/2}$$
$$=(\underline{\underline{L}}\cdot\underline{\underline{C}})^{-1/2}\cdot\underline{\underline{L}}=\underline{\underline{L}}\cdot(\underline{\underline{C}}\cdot\underline{\underline{L}})^{-1/2}. \tag{56}$$

In the special case that $\underline{\underline{L}}$ and $\underline{\underline{C}}$ commute we have

$$\underline{\underline{Z}}=\underline{\underline{L}}^{1/2}\cdot\underline{\underline{C}}^{-1/2}=\underline{\underline{C}}^{-1/2}\cdot\underline{\underline{L}}^{1/2}. \tag{57}$$

Because $\underline{\underline{L}}$ and $\underline{\underline{C}}$ are symmetric, from (55) we find that $\underline{\underline{Z}}$ and $\underline{\underline{Y}}$ are also symmetric. Equation (56) obviously generalizes the familiar formula $Z_c=\sqrt{L/C}$ of the common coaxial line. The result (55) was obtained earlier in [5] and (56) in [6].

VIII. Different Eigenvalue Problems

A mode in a multiconductor line can be defined as a set of voltage and current vectors $\underline{U}^j$, $\underline{I}^j(j=1\cdots N)$ in terms of which any voltage and current distribution $\underline{U}$, $\underline{I}$ can be expressed as a linear combination. The propagating quasi-TEM modes can be defined as those voltage distributions which are propagated along the line and changed only by a scalar factor: $\underline{U}(z)=\kappa(z)\underline{U}(0)$. Inserting in (51), we see that $\kappa(z)$ must be an eigenvalue of $\exp(-j\omega(\underline{\underline{L}}\cdot\underline{\underline{C}})^{1/2}z)$ and $\underline{U}(0)=\underline{U}_p$ the corresponding eigenvector. Hence, κ must be of the exponential form $\exp(-j\beta z)$ and β satisfies the eigenvalue equation

$$\left[(\underline{\underline{L}}\cdot\underline{\underline{C}})^{1/2}-\frac{\beta}{\omega}\underline{\underline{I}}\right]\cdot\underline{U}_p=0. \tag{58}$$

The current vector $\underline{I}_p$ must satisfy the corresponding equation

$$\left[(\underline{\underline{C}}\cdot\underline{\underline{L}})^{1/2}-\frac{\beta}{\omega}\underline{\underline{I}}\right]\cdot\underline{I}_p=0. \tag{59}$$

It is easy to see that the eigenvalues of both (58) and (59) are the same, or for a voltage mode there corresponds a current mode, which propagates with the same factor β. The eigenvectors $\underline{U}_p$ and $\underline{I}_p$ are different, in general, so that for a propagating mode there does not exist a scalar impedance of the form $\underline{U}_p^j=Z^j\underline{I}_p^j$. We must, then, write in the general matrix form $\underline{U}_p^j=\underline{\underline{Z}}\cdot\underline{I}_p^j$, and, in general, may state that if a mode has a scalar propagation factor, it does not have a scalar impedance.

Studying the impedance eigenvalue problem

$$(\underline{\underline{Z}}-z\underline{\underline{I}})\cdot\underline{I}_i=0 \tag{60}$$

or

$$\left[(\underline{\underline{C}}\cdot\underline{\underline{L}})^{1/2}-z\underline{\underline{C}}\right]\cdot\underline{I}_i=0 \tag{61}$$

and the corresponding admittance problem

$$(\underline{\underline{Y}}-y\underline{\underline{I}})\cdot\underline{U}_i=0 \tag{62}$$

or

$$\left[(\underline{\underline{L}}\cdot\underline{\underline{C}})^{1/2}-y\underline{\underline{L}}\right]\cdot\underline{U}_i=0 \tag{63}$$

we in fact find that they are different from those of (58) and (59). Thus the eigenvectors $\underline{U}_i^j$, $\underline{I}_i^j (j=1\cdots N)$ do not coincide with $\underline{U}_p^j$, $\underline{I}_p^j$ in general. We could call these solutions impedance modes, because for them we have a scalar impedance but generally the propagation factor is a matrix, i.e., the voltage and current distributions change along the line. The eigenvectors of (60) and (62) are the same and the eigenvalues are related by $z^j=1/y^j$, as is easily seen.

The impedance modes are important for the following reasons. Firstly, in a homogeneous multiconductor line the propagating modes are degenerate and no preferred system $\underline{U}_p^j$, $\underline{I}_p^j$ exists, whereas the impedance modes form a natural nondegenerate set of basis vectors. Secondly, while being not attractive for propagation considerations, the impedance modes are very useful at the terminal planes of the multiconductor line. For example, every impedance mode sees its own scalar characteristic impedance z^j of a nonterminating line.

Further, we may consider the static eigenvalue problems

$$(\underline{\underline{C}}-c\underline{\underline{I}})\cdot\underline{U}_s=0 \tag{64}$$

$$(\underline{\underline{L}}-l\underline{\underline{I}})\cdot\underline{I}_s=0 \tag{65}$$

which relate the charge with the voltage by $\underline{Q}_s=c\underline{U}_s$ and the magnetic flux with the current by $\underline{\Psi}_s=l\underline{I}_s$. The eigenvectors $\underline{U}_s^j$, $\underline{I}_s^j (j=1\cdots N)$ may be called the static modes and they again are different from the propagating modes and impedance modes, in general. Expanding the voltage distribution $\underline{U}$ in terms of $\underline{U}_s^j$ leads to N distinct scalar electrostatic problems each with a scalar capacitance c^j, and the total charge distribution is obtained in the form $\underline{Q}=\Sigma c^j\underline{U}_s^j$. The static modes do not seem to be applicable to propagation problems in general. In special cases they are, however, as will be seen.

IX. Propagating Impedance Modes

Finally we study under which circumstances do the propagating modes have a scalar impedance, or, what is equivalent, when do impedance modes have a scalar propagation factor. Because in this case, the eigenvectors $\underline{U}_p^j$ and $\underline{I}_p^j$ are the same, from (58) and (59) we see that this is only possible if we have

$$(\underline{\underline{L}}\cdot\underline{\underline{C}})^{1/2}=(\underline{\underline{C}}\cdot\underline{\underline{L}})^{1/2} \tag{66}$$

or $\underline{\underline{L}}$ and $\underline{\underline{C}}$ must commute: $\underline{\underline{L}}\cdot\underline{\underline{C}}=\underline{\underline{C}}\cdot\underline{\underline{L}}$. This implies that, from the symmetry properties of $\underline{\underline{L}}$ and $\underline{\underline{C}}$, the matrix $(\underline{\underline{L}}\cdot\underline{\underline{C}})^{1/2}$ must be symmetric. Further, it is known from matrix algebra [3] that two semisimple matrices (like the real and symmetric matrices $\underline{\underline{L}}$, $\underline{\underline{C}}$) commute only when they possess the same set of eigenvectors. Hence, $\underline{\underline{L}}$ and $\underline{\underline{C}}$ and also $(\underline{\underline{L}}\cdot\underline{\underline{C}})^{1/2}$ and $\underline{\underline{Z}}=\underline{\underline{C}}^{-1}\cdot(\underline{\underline{C}}\cdot\underline{\underline{L}})^{1/2}$ must all possess the same set of eigenvectors for the propagating modes to have a scalar impedance. In this case it is possible to express the eigenvalues β^j, z^j in terms of the static eigenvalues c^j, l^j

$$\beta^j=\omega\sqrt{l^jc^j} \tag{67}$$

$$z^j=\sqrt{l^j/c^j}=1/y^j. \tag{68}$$

Conversely, if the static problems (64) and (65) have the same set of eigenvectors, the propagating modes always have a scalar impedance. Thus the problem reduces to two static problems and the relation between them. It is evident that, since changing the function $\mu(\bar{\rho})$ only changes $\underline{\underline{L}}$ and not $\underline{\underline{C}}$, commutativity is indeed a special case and occurs only in case of some symmetrics or conditions for the medium.

If we require that both static eigenvalue problems possess the same set of eigenvectors with no conditions to the positions and form of the conductors, we evidently end up in the condition for the medium $\mu\varepsilon=$constant. In this case we have $\underline{\underline{L}}\cdot\underline{\underline{C}}=\mu\varepsilon\underline{\underline{I}}$ so that any vectors $\underline{U}$, $\underline{I}$ are eigenvectors of (58) and (59). This means that the propagating mode eigenvalue problem is degenerate whereas the impedance mode eigenvalue problem is not. All vectors do not satisfy (60)–(65), or all propagating modes are not impedance modes, but impedance modes are propagating modes.

When certain symmetries are satisfied, the commutation of $\underline{\underline{L}}$ and $\underline{\underline{C}}$ is fulfilled. For example, for a symmetric coupled microstrip line we have $L_{11}=L_{22}$ and $C_{11}=C_{22}$ which results in $\underline{\underline{L}}\cdot\underline{\underline{C}}=\underline{\underline{C}}\cdot\underline{\underline{L}}$, as is easy to confirm. The symmetric and antisymmetric modes are propagating impedance modes, whereas for an asymmetric microstrip the propagating modes do not possess scalar impedances. More generally, the symmetries are expressed in terms of symmetry matrices $\underline{\underline{S}}_i$, which may change the indices of the conductors. If such an operation is performed on the shielded N-line, it remains invariant if the corresponding symmetry exists. Hence, we must have the properties [4]

$$\underline{\underline{S}}_i\cdot\underline{\underline{L}}\cdot\underline{\underline{S}}_i^{-1}=\underline{\underline{L}} \qquad \underline{\underline{S}}_i\cdot\underline{\underline{C}}\cdot\underline{\underline{S}}_i^{-1}=\underline{\underline{C}} \tag{69}$$

if the matrices $\underline{\underline{C}}$, $\underline{\underline{L}}$ do not change in the symmetry change of indices.

It is thus seen that if a symmetry matrix exists, both $\underline{\underline{L}}$ and $\underline{\underline{C}}$ commute with this matrix. If the symmetry matrix possesses at least one nondegenerate eigenvalue, it follows that the eigenvector corresponding to this eigenvalue is also an eigenvalue of both $\underline{\underline{L}}$ and $\underline{\underline{C}}$. Thus this eigenvector is a solution to all (58)–(65) and it is a propagating impedance mode with a propagation factor and an impedance as in (67) and (68). If the symmetry matrix possesses more nondegenerate eigenvalues, for every eigenvalue there thus exists one propagating impedance mode. Commutativity of the matrices $\underline{\underline{L}}$ and $\underline{\underline{C}}$ occurs for complete symmetry of the multiconductor line, i.e., either there are N distinct eigenvalues for the symmetry matrix $\underline{\underline{S}}$ or there exist several symmetry matrices $\underline{\underline{S}}_i$ with the total of N nondegenerate eigenvalues, each corresponding to a propagating impedance mode of the multiconductor line.

Acknowledgment

The idea of generalizing the properties of inhomogeneous lines to inhomogeneous multiconductor lines was originated by K. Mannersalo.

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Equivalent Circuits of Binomial Form Nonuniform Coupled Transmission Lines

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Abstract—**Equivalent circuits of nonuniform coupled transmission lines whose self and mutual characteristic admittance distributions obey binomial form are presented. Telegrapher's equations of these nonuniform coupled transmission lines can be solved exactly using Bessel functions of fractional order. By decomposing the chain matrix, it is shown that equivalent circuits of these nonuniform coupled transmission lines consist of cascade connections of lumped reactance elements, uncoupled uniform transmission lines and ideal transformers.**

I. Introduction

COUPLED TRANSMISSION lines are very important in microwave network theory. They are used extensively in all types of microwave components: filters, couplers, matching sections, and equalizers. Uniform coupled transmission lines have been described by many authors [1]–[15], and it is well known that equivalent representations of coupled transmission lines are very significant techniques in the analysis and synthesis. Nonuniform coupled transmission lines show good transmission responses and may also be important in microwave network theory. In general, the analysis of nonuniform coupled transmission lines becomes hard work because of difficulty of finding exact network functions. The analysis of particular nonuniform coupled transmission lines, for instance, exponential or hyperbolic tapered coupled transmission lines, have been reported [16], [17], but useful equivalent representations have not been obtained.

In this paper, we investigate equivalent circuits of nonuniform coupled transmission lines whose self and mutual characteristic admittance distributions obey binomial form. First, it is shown that telegrapher's equations of these nonuniform coupled transmission lines can be solved exactly using Bessel functions of fractional order. Then, by decomposing chain matrices of these circuits, we can show that equivalent circuits of these nonuniform coupled transmission lines are expressed as cascade connections of lumped reactance elements, uncoupled uniform transmission lines and ideal transformers. Two-port equivalent circuits of parabolic tapered coupled transmission lines with appropriate terminal conditions imposed are also presented by using equivalent representations shown in this paper.

Manuscript received November 25, 1980; revised March 2, 1981.
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II. Exact Solutions of Telegrapher's Equations

The $2n$th-order binomial form coupled transmission lines (BFCTL) are nonuniform coupled transmission lines whose self and mutual characteristic admittance distributions are given as the binomial form $(ax+b)^{\pm 2n}$, where x is the distance along the line, a and b are constants and n is an integer. The lossless $2n$th-order BFCTL above a ground

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